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Can Harmonic Functions Constitute Closed-Form Buckling Modes of Inhomogeneous Columns?

Ivo Caliò*
University of Catania, 95125 Catania, Italy
and
Isaac Elishakoff†
Florida Atlantic University, Boca Raton, Florida 33431

It is uniformly known that the buckling modes of uniform columns are given by trigonometric, namely, harmonic, functions. For inhomogeneous columns the buckling modes usually are derived via special functions including Bessel and Lommel functions. Recently it was demonstrated that the buckling modes of specific inhomogeneous columns assume a simple polynomial form. The question posed in the title of this study therefore naturally arises. It is shown that the reply to this query is affirmative. Four cases of harmonically varying buckling modes are postulated and semi-inverse problems are solved that result in the distributions of the flexural rigidity compatible to the preselected modes and to specified axial load distributions. In all cases the closed-form solutions are obtained for the eigenvalue parameter.

Nomenclature

A_j	=	constant coefficients [defined in Eq. (4)]
a	=	real number [defined in Eq. (1)]
$D(x)$	=	axially varying flexural rigidity $E(x)I(x)$
$E(x)$	=	modulus of elasticity
I	=	positive constant [defined in Eq. (1)]
$I(x)$	=	moment of inertia
L	=	length of the column
$N(x)$	=	axial compressive load distribution
P	=	critical load
$p(x)$	=	distributed axial load
q	=	eigenvalue
$w(x)$	=	displacements
x	=	axial coordinate
α	=	real number [defined in Eq. (4)]
β	=	ratio A_1/A_0
ξ	=	nondimensional coordinate
$\psi(x)$	=	specified harmonic buckling mode

Introduction

RECENTLY several studies have been published that reported novel closed-form solutions for columns of variable flexural rigidity. Elishakoff and Rollot[1] utilized computerized symbolic algebra to derive several closed-form solutions. A systematic derivation has been conducted by Elishakoff[2,3] and Guede and Elishakoff[4] in which polynomial mode shapes with closed-form solutions for the buckling loads have been derived. Namely, Elishakoff's study[2] dealt with generalized Euler column, whereas in the paper[3] Euler's column under axial concentrated loads have been considered. Guede and Elishakoff[4] generalized this solution for the vibrating column.

It is of interest to extend the class of functions acceptable for the buckling modes, because the polynomials have been studied before. Here we consider the trigonometric mode shapes, as candidate functions for the buckling modes.

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*Assistant Professor, Dipartimento di Ingegneria Civile e Ambientale; icalio@dica.unict.it.

†Professor, Department of Mechanical Engineering; ielishak@me.fau.edu.

We found in the literature a reference on a single work, by Dinnik,[5] as quoted in the selected works by Dinnik[6] whose title suggests that the axial variation of the cross section or of the flexural rigidity was sinusoidal. (We were unable to obtain the paper itself.) Another book by Dinnik[7] references work by Bleich,[8] who considered by approximate method buckling of columns whose moment of inertia varies as follows:

$$I(x) = I[\sin(a+x)/(a+L)\pi]^2 \tag{1}$$

Dinnik[7] mentions that "the method and the numerical coefficients obtained by Bleich[8] call for doubts in their correctness." Hence Dinnik[7] reobtained the formulas by numerical integration.

In this study, instead of polynomial buckling modes trigonometric functions are postulated. It turns out that for specialized distributions of axial loads the flexural rigidity is also a harmonic function that is found by satisfying the boundary conditions and the governing differential equation.

The determined buckling loads have been contrasted with the results obtained by energy method.

Formulation of the Problem

The differential equation that governs the buckling of the nonuniform column under a prescribed distributed axial load is

$$\frac{d^2}{dx^2}\left[D(x)\frac{d^2w}{dx^2}\right] + \frac{d}{dx}\left[qN(x)\frac{dw}{dx}\right] = 0 \tag{2}$$

where $N(x)$ depends on the particular spatial form of the distributed axial load $p(x) = dN/dx$.

In this paper the differential equation (2) will be solved for four sets of boundary conditions corresponding to the following columns: 1) simply supported at its both ends, 2) clamped at one end and free at the other, 3) simply supported at one end and guided at the other, and 4) clamped at both ends. For simplicity, the nondimensional coordinate $\xi = x/L$ is introduced. Then the differential equation becomes

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2w}{d\xi^2}\right] + \frac{d}{d\xi}\left[qL^3N(\xi)\frac{dw}{d\xi}\right] = 0 \tag{3}$$

The semi-inverse problem is posed as follows: Find an inhomogeneous column with a specified harmonic buckling mode $w(\xi) = \psi(\xi)$ that satisfies the boundary conditions and the differential equation (2). This problem requires the determination of the distribution of stiffness $D(\xi)$ that together with the specific distribution of axial load and the prescribed buckling mode satisfy the governing eigenvalue problem.

In recent studies the polynomial buckling modes have been postulated to solve the problem at hand. Here the objective is to widen the class by including trigonometric functions as buckling modes. These appear to be suitable candidate functions, because for the uniform column under axial compression by a concentrated axial load the sinusoidal function serves as a buckling mode.

The flexural rigidity $D(\xi)$ is represented as follows:

$$D(\xi) = A_0 + A_1 \sin(\alpha\pi\xi) + A_2 \cos(\alpha\pi\xi) \tag{4}$$

The semi-inverse problem might have no solution or multiple solutions or a unique solution. It will be shown that for a specified distribution of axial load the solution turns out to be unique; moreover, a single constant will be needed only.

Column Simply Supported at Both Ends

Let us consider a simply supported column subjected to a distributed axial load $p(\xi)$, which leads to distribution of axial force $N(\xi)$. The governing differential equation of the column reads

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2 w}{d\xi^2}\right] + \frac{d}{d\xi}\left[qL^3 N(\xi)\frac{dw}{d\xi}\right] = 0 \tag{5}$$

The following harmonic representation of the stiffness is introduced:

$$D(\xi) = A_0 + A_1 \sin(\pi\xi) \tag{6}$$

In addition the buckling mode of the homogeneous column under constant axial force

$$\psi(\xi) = \sin(\pi\xi) \tag{7}$$

is postulated. Bearing in mind the expressions (6) and (7), the first term in the differential equation can be rewritten as

$$-\pi^2\frac{d^2}{d\xi^2}\left[A_0\sin(\pi\xi) + A_1\sin^2(\pi\xi)\right]$$

$$= -\pi^3\frac{d}{d\xi}[A_0\cos(\pi\xi) + 2A_1\sin(\pi\xi)\cos(\pi\xi)] \tag{8}$$

whereas the second term assumes the expression

$$\frac{d}{d\xi}[qL^3 N(\xi)\pi\cos(\pi\xi)] \tag{9}$$

In view of expressions (8) and (9), the differential equation (5) takes the form

$$\frac{d}{d\xi}\left\{\left[\frac{qL^3}{\pi^2}N(\xi) - A_0 - 2A_1\sin(\pi\xi)\right]\cos(\pi\xi)\right\} = 0 \tag{10}$$

Equation (10) is naturally satisfied for all of the distributions of axial force $N(\xi)$ that are proportional to

$$N(\xi) \propto A_0 + 2A_1\sin(\pi\xi) \tag{11}$$

We consider the following particular cases:

Case 1: $A_0 = 0; A_1 \neq 0$

In this case the stiffness distribution is

$$D(\xi) = A_1\sin(\pi\xi) \tag{12}$$

Then, assuming $N(\xi) = \sin(\pi\xi)$, the following eigenvalue is obtained:

$$q = 2A_1\pi^2/L^3 \tag{13}$$

Case 2: $A_0 \neq 0; A_1 = 0$

This case corresponds to the homogeneous column

$$D(\xi) = A_0 \tag{14}$$

Then, assuming $N(\xi) = P =$ constant, the nondimensional equilibrium differential equation can be written in the form

$$A_0\frac{d^4\psi}{d\xi^4} + PL^2\frac{d^2\psi}{d\xi^2} = 0 \tag{15}$$

from which the well-known critical load of the homogeneous column under constant axial load is obtained:

$$P = A_0\pi^2\Big/L^2 \tag{16}$$

General Case: $A_0 \neq 0; A_1 \neq 0$

This case corresponds to the inhomogeneous column with stiffness distribution

$$D(\xi) = A_0[1 + \beta\sin(\pi\xi)] \tag{17}$$

Considering that the stiffness must be positive, the following inequality must be verified:

$$A_0 > 0, \qquad \beta > -1$$

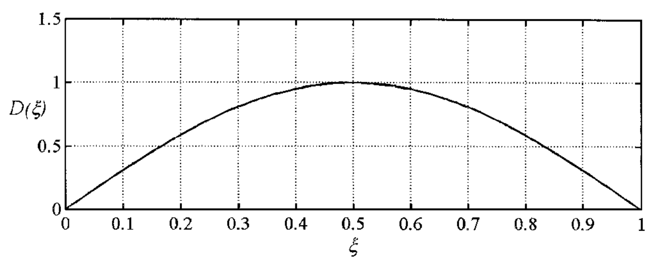
Furthermore, considering that the axial load must be positive and proportional to $N(\xi) \propto 1 + 2\beta\sin(\pi\xi)$, the ratio β must also satisfy the inequality

$$\beta \geq -\tfrac{1}{2}$$

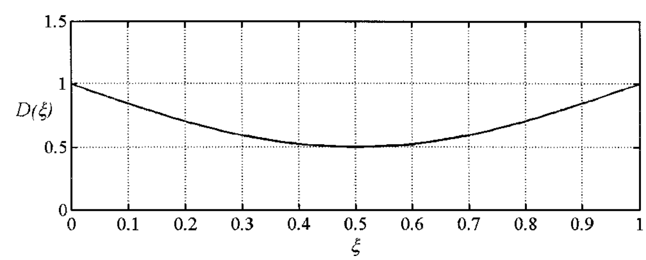
Assuming $N(\xi) = 1 + 2\beta\sin(\pi\xi)$, which can be associated to a unit concentrated axial load plus a distributed axial load $p(\xi) = -2\beta\pi L\cos(\pi\xi)$, the following eigenvalue is obtained:

$$q = A_0\pi^2\Big/L^3 \tag{18}$$

In Fig. 1 the stiffness distributions relative to the limit case with $A_0 = 0$ and $A_1 = 1$ and to the case with $A_0 = 1$ and $\beta = -0.5$ are reported.



a) $A_0 = 0; A_1 = 1$



b) $A_0 = 1; \beta = -0.5$

Fig. 1 Flexural rigidity for the column that is simply supported at both ends.

Column Clamped at One End and Free at the Other

For a cantilever column an harmonic function that satisfies the boundary conditions is

$$\psi(\xi) = 1 - \cos[(\pi/2)\xi] \tag{19}$$

Employing a trigonometric representation of the flexural rigidity as follows:

$$D(\xi) = A_0 + A_1 \cos[(\pi/2)\xi] \tag{20}$$

the first term in the differential equation (3) can be rewritten as

$$\frac{\pi^3}{8}\frac{d}{d\xi}\left\{\left[-A_0 - 2A_1 \cos\left(\frac{\pi}{2}\xi\right)\right]\sin\left(\frac{\pi}{2}\xi\right)\right\} \tag{21}$$

whereas the second term assumes the expression

$$\frac{\pi}{2}\frac{d}{d\xi}\left[qL^3 N(\xi)\sin\left(\frac{\pi}{2}\xi\right)\right] \tag{22}$$

as a consequence the differential equation (5) can be written as

$$\frac{d}{d\xi}\left\{\left[\frac{4}{\pi^2}qL^3 N(\xi) - A_0 - 2A_1\cos\left(\frac{\pi}{2}\xi\right)\right]\sin\left(\frac{\pi}{2}\xi\right)\right\} = 0 \tag{23}$$

It is easy to recognize that all of the distributions of axial force $N(\xi)$ that are proportional to

$$N(\xi) \propto A_0 + 2A_1 \cos[(\pi/2)\xi] \tag{24}$$

satisfy the equilibrium differential equation. We consider the following particular cases:

Case 1: $A_0 = 0; A_1 \neq 0$

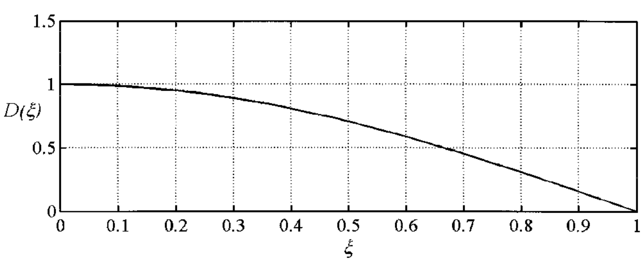
In this case the stiffness distribution is

$$D(\xi) = A_1 \cos[(\pi/2)\xi] \tag{25}$$

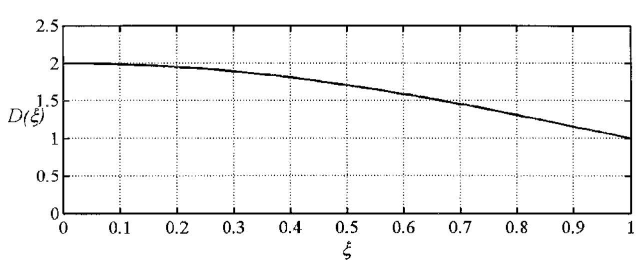
Then assuming $N(\xi) = \cos[(\pi/2)\xi]$ the following eigenvalue is obtained:

$$q = A_1\pi^2 \big/ 2L^3 \tag{26}$$

In Fig. 2a the stiffness distribution is reported with reference to $A_0 = 0$ and $A_1 = 1$.



a) $A_0 = 0; A_1 = 1$



b) $A_0 = 1; \beta = 1$

Fig. 2 Flexural rigidity for the column that is clamped at one end and free at the other.

Case 2: $A_0 \neq 0; A_1 = 0$

This case corresponds to the homogeneous column

$$D(\xi) = A_0 \tag{27}$$

Then, assuming $N(\xi) = P$ and $q = 1$, the critical load of the homogeneous cantilever beam is obtained:

$$P = A_0\pi^2 \big/ 4L^2 \tag{28}$$

General Case: $A_0 \neq 0; A_1 \neq 0$

This case corresponds to the inhomogeneous column with stiffness distribution

$$D(\xi) = A_0\{1 + \beta\cos[(\pi/2)\xi]\} \tag{29}$$

Assuming $N(\xi) = \{1 + 2\beta\cos[(\pi/2)\xi]\}$ the following eigenvalue is obtained:

$$q = A_0\pi^2 \big/ 4L^3 \tag{30}$$

Also in this case the stiffness constants A_0 and β must satisfy the following inequalities:

$$A_0 > 0, \qquad \beta \geq -\tfrac{1}{2}$$

In Fig. 2b the stiffness distribution corresponding to the values of the stiffness coefficients $A_0 = \beta = 1$ is reported.

Verification of the Natural Boundary Conditions

For this particular case the natural boundary conditions at the free end need to be verified because they are not obvious. The following boundary conditions at the free end must be verified:

$$D(\xi)\psi''(\xi) = 0 \tag{31}$$

$$[D(\xi)\psi''(\xi)]' + qN(\xi)\psi'(\xi) = 0 \tag{32}$$

Condition (31) is met because $\psi''(\xi)$ is zero at the free end. In the following the condition (32) will be verified for case 1 relative to the distributed load; the verification relative to the general case is similar. The critical load is $q = A_1\pi^2/2L^3$. Therefore, boundary condition (32) becomes

$$A_1(\pi^3/4)\cos[(\pi/2)\xi]\sin[(\pi/2)\xi] + (A_1/2)(\pi^2/L^3) \times \cos[(\pi/2)\xi](\pi/2)\sin[(\pi/2)\xi] = 0 \qquad \forall\xi \tag{33}$$

Column Simply Supported at One End and Guided at the Other

For a column that is simply supported at one end and guided at the other, the desired buckling mode is taken as

$$\psi(\xi) = \sin[(\pi/2)\xi] \tag{34}$$

This mode shape coincides with the exact buckling mode for the uniform column under axial constant compression.

Considering a simple trigonometric representation of the flexural rigidity as follows:

$$D(\xi) = A_0 + A_1 \sin[(\pi/2)\xi] \tag{35}$$

The first term of the equilibrium differential equation can be written in the following form:

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2\psi}{d\xi^2}\right] = -\frac{\pi^3}{8}\frac{d}{d\xi}\left\{\left[-A_0 - 2A_1\sin\left(\frac{\pi}{2}\xi\right)\right]\cos\left(\frac{\pi}{2}\xi\right)\right\} \tag{36}$$

whereas the second term assumes the expression

$$\frac{\pi}{2}\frac{d}{d\xi}\left[qL^3N(\xi)\cos\left(\frac{\pi}{2}\xi\right)\right] \tag{37}$$

and the equilibrium differential equation (5) can be written as

$$\frac{d}{d\xi}\left\{\left[\frac{4}{\pi^2}qL^3N(\xi)-A_0-2A_1\sin\left(\frac{\pi}{2}\xi\right)\right]\cos\left(\frac{\pi}{2}\xi\right)\right\}=0 \tag{38}$$

As a consequence the class of distributions of axial force

$$N(\xi)\propto A_0+2A_1\sin[(\pi/2)\xi] \tag{39}$$

satisfies the equilibrium differential equation. The following particular cases are considered:

Case 1: $A_0=0; A_1\neq 0$

In this case the stiffness distribution is

$$D(\xi)=A_1\sin[(\pi/2)\xi] \tag{40}$$

Then, assuming $N(\xi)=\sin[(\pi/2)\xi]$ the following eigenvalue is obtained:

$$q=A_1\pi^2\big/2L^3 \tag{41}$$

The stiffness distribution corresponding to $A_0=0$; $A_1=1$ is reported in Fig. 3a.

Case 2: $A_0\neq 0; A_1=0$

This case corresponds to the homogeneous column $D(\xi)=A_0$; assuming $N(\xi)=P$ and $q=1$, the critical load of the homogeneous simply supported guided column is obtained:

$$P=A_0\pi^2\big/4L^2 \tag{42}$$

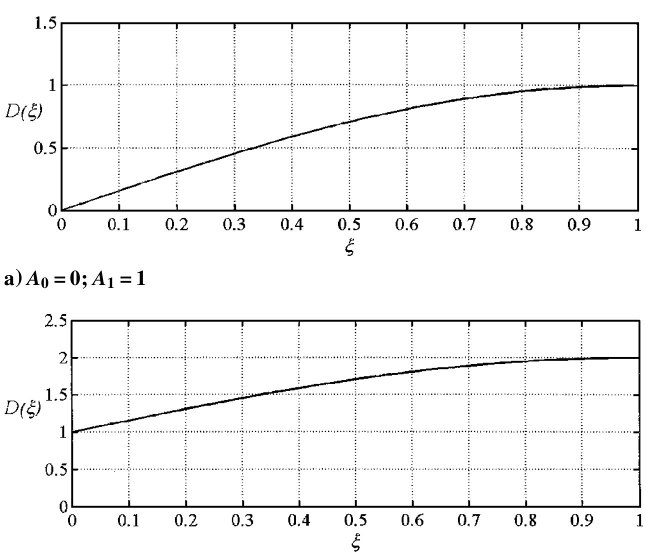
General Case: $A_0\neq 0; A_1\neq 0$

This case corresponds to the inhomogeneous column with stiffness distribution

$$D(\xi)=A_0\{1+\beta\sin[(\pi/2)\xi]\} \tag{43}$$

Assuming $N(\xi)=[1+2\beta\sin\pi/2\xi)]$, the following eigenvalue is obtained:

$$q=A_0\pi^2\big/4L^3 \tag{44}$$



a) $A_0=0; A_1=1$

b) $A_0=1; \beta=1$

Fig. 3 Flexural rigidity for the column that is simply supported at one end and guided at the other.

The stiffness and the axial load must be nonnegative; therefore, the following inequalities must be verified:

$$A_0>0,\qquad \beta>-\tfrac{1}{2}$$

In Fig. 3b the stiffness distribution corresponding to $A_0=\beta=1$ is displayed. It is remarkable that the expressions (40) and (44) coincide formally with their counterparts (30) and (26) for the cantilever column.

Column Clamped at Both Its Ends

Let us consider a column clamped at both its ends. The boundary conditions are satisfied for the following buckling mode:

$$\psi(\xi)=1-\cos(2\pi\xi) \tag{45}$$

Assuming the following trigonometric representation of the flexural rigidity:

$$D(\xi)=A_0+A_1[1-\cos(2\pi\xi)] \tag{46}$$

after algebraic manipulation the first term of the equilibrium differential equation can be written as

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2\psi}{d\xi^2}\right]=8\pi^3\frac{d}{d\xi}(\{-A_0-A_1[1-2\cos(2\pi\xi)]\}\sin(2\pi\xi)) \tag{47}$$

and the second term assumes the expression

$$2\pi\frac{d}{d\xi}[qL^3N(\xi)\sin(2\pi\xi)] \tag{48}$$

Therefore, considering expressions (47) and (48) the differential equation (5) can be put as

$$\frac{d}{d\xi}\left(\left\{\frac{1}{4\pi^2}qL^3N(\xi)-A_0-A_1[1-2\cos(2\pi\xi)]\right\}\sin(2\pi\xi)\right)=0 \tag{49}$$

from which it is apparent that the distributions of axial force $N(\xi)$ that are proportional to

$$N(\xi)\propto A_0+A_1[1-2\cos(2\pi\xi)] \tag{50}$$

satisfy the equilibrium differential equation for a specified value of the buckling load.

Bearing in mind that $N(\xi)$ must be positive, the case $A_0=0$; $A_1\neq 0$ must be discarded. The following particular cases are considered.

Case 1: $A_0\neq 0; A_1=0$

This case corresponds to the homogeneous column $D(\xi)=A_0$. Assuming a constant axial force P and $q=1$, the critical load of the homogeneous column is obtained:

$$P=4A_0\pi^2\big/L^2 \tag{51}$$

General Case: $A_0\neq 0; A_1\neq 0$

This case corresponds to the inhomogeneous column with stiffness distribution

$$D(\xi)=A_0\{1+\beta[1-\cos(2\pi\xi)]\} \tag{52}$$

Assuming $N(\xi)=1+\beta[1-2\cos(2\pi\xi)]$, the following eigenvalue is obtained:

$$q=4A_0\pi^2\big/L^3 \tag{53}$$

Observing that the stiffness and the axial load cannot be negative, the following inequalities must be verified:

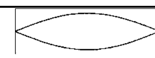
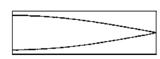
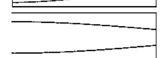
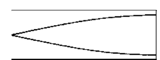
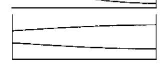
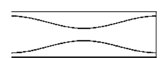
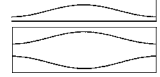
$$A_0>0,\qquad -\tfrac{1}{3}<\beta\leq 1$$

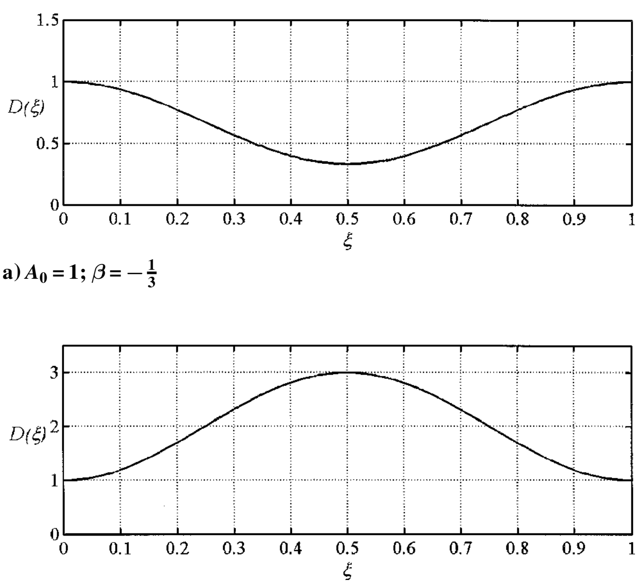
The stiffness distributions relative to the case corresponding to $A_0=1$ and $\beta=-\frac{1}{3}$ and to the limit case $A_0=1$ and $\beta=1$ are reported in Figs. 4a and 4b.

Table 1 Approximate results

Boundary condition and trial function	Stiffness distribution $D(\xi)$	Axial load distribution $N(\xi)$	Rayleigh's quotient Eq. (54)
S-S (simply supported)	$A_1 \sin(\pi\xi)$	$\sin(\pi\xi)$	$19.7541\dfrac{A_1}{L^3}$
$\xi - 2\xi^3 + \xi^4$	$A_0[1 + \beta \sin(\pi\xi)]$	$1 + 2\beta \sin(\pi\xi)$	$\left(9.877 + \dfrac{0.006339}{1.196 + \beta}\right)\dfrac{A_0}{L^3}$
C-F (clamped–free)	$A_1 \cos\left(\dfrac{\pi}{2}\xi\right)$	$\cos\left(\dfrac{\pi}{2}\xi\right)$	$5.406\dfrac{A_1}{L^3}$
$6\xi^2 - 4\xi^3 + \xi^4$	$A_0\left[1 + \beta \cos\left(\dfrac{\pi}{2}\xi\right)\right]$	$\left[1 + 2\beta \cos\left(\dfrac{\pi}{2}\xi\right)\right]$	$\left(2.703 + \dfrac{0.0992}{1.0235 + \beta}\right)\dfrac{A_0}{L^3}$
S-G (simply guided)	$A_1 \sin\left(\dfrac{\pi}{2}\xi\right)$	$\sin\left(\dfrac{\pi}{2}\xi\right)$	$4.988\dfrac{A_1}{L^3}$
$\dfrac{3}{2}\xi - \dfrac{1}{2}\xi^3$	$A_0\left[1 + \beta \sin\left(\dfrac{\pi}{2}\xi\right)\right]$	$1 + 2\beta \sin\left(\dfrac{\pi}{2}\xi\right)$	$\left(2.494 + \dfrac{0.00643}{1.129 + \beta}\right)\dfrac{A_0}{L^3}$
C-C (clamped–clamped) $\xi^2 - 2\xi^3 + \xi^4$	$A_0\{1 + \beta[1 - \cos(2\pi\xi)]\}$	$1 + \beta[1 - 2\cos(2\pi\xi)]$	$\left(43.88 - \dfrac{7.33412}{3.900 + \beta}\right)\dfrac{A_0}{L^3}$

Table 2 Comparison of the closed-form solution with the approximate results

Boundary condition	Considered case	Stiffness distribution	Closed-form solution	Rayleigh's quotient	Difference with closed-form solution, %	Shape of a column with circular cross section
S-S	1	$A_0 = 0;\ A_1 \neq 0$	$2\pi^2(A_1/L^3)$	$19.754(A_1/L^3)$	0.076	
	General	$A_0 \neq 0;\ A_1 = -\frac{1}{2}A_0$	$\pi^2(A_0/L^3)$	$9.886(A_0/L^3)$	0.172	
C-F	1	$A_0 = 0;\ A_1 \neq 0$	$\pi^2/2(A_1/L^3)$	$5.406(A_1/L^3)$	9.566	
	General	$A_0 \neq 0;\ A_1 = A_0$	$\pi^2/4(A_0/L^3)$	$2.7521(A_0/L^3)$	10.442	
S-G	1	$A_0 = 0;\ A_1 \neq 0$	$\pi^2/2(A_1/L^3)$	$4.988(A_1/L^3)$	1.094	
	General	$A_0 \neq 0;\ A_1 = A_0$	$\pi^2/4(A_0/L^3)$	$2.497(A_0/L^3)$	1.216	
C-C	General	$A_0 \neq 0;\ A_1 = -\frac{1}{3}A_0$	$4\pi^2(A_1/L^3)$	$41.824(A_1/L^3)$	5.942	
	General	$A_0 \neq 0;\ A_1 = A_0$	$4\pi^2(A_0/L^3)$	$42.383(A_0/L^3)$	7.358	



a) $A_0 = 1$; $\beta = -\frac{1}{3}$

b) $A_0 = 1$; $\beta = 1$

Fig. 4 Flexural rigidity for the column clamped at both its ends.

Discussion

It appears to be instructive to list also the buckling loads obtained by some other methods. To compare the obtained analytical results with those obtained considering approximate solutions, we also conducted a calculation by the energy method using the Rayleigh quotient

$$q = \frac{1}{L^3}\frac{\int_0^1 D(\xi)[\psi''(\xi)]^2\,\mathrm{d}\xi}{\int_0^1 N(\xi)[\psi'(\xi)]^2\,\mathrm{d}\xi} \tag{54}$$

For the trial function polynomial expressions were used for simplicity. The approximate results are reported in Table 1. In the general case the approximate solutions are dependent on both the parameters A_0 and β, whereas the closed-form solutions depend on only one parameter. The true mode shapes yield buckling loads slightly lower than the approximate ones, whose values are reported in the last column of Table 1, for any admissible value of the parameter β.

The comparison of the closed-form solution with the approximate results, for all of the cases considered earlier, is reported in Table 2. In the last column of the same table are also sketched the shapes of corresponding columns with circular cross sections.

The solutions reported herein can be used as benchmark problems. Also, at the time when the technology is available to produce an arbitrary distribution of elastic modulus along the axis of the column, one will be able to design inhomogeneous columns with preselected buckling loads. For example, let us consider the case when we need to design a column that is simply supported at its ends and the design requirement consists of the demand that the buckling load be in excess of a preselected number δ. Then Eq. (18) yields the following value of A_0:

$$A_0 = \delta L^3/\pi^2 \tag{55}$$

This means that if the flexural rigidity is

$$D(\xi) = \delta L^3/\pi^2[1 + \beta \sin(\pi\xi)] \tag{56}$$

with $\beta > -1$ and the external distribution of axial load $p(\xi) = -\delta[2\beta\pi L \cos(\pi\xi)]$ then the buckling load distribution will precisely coincide with δ. Likewise, for the other boundary conditions the designed flexural rigidities read

$$D(\xi) = (4\delta L^3/\pi^2)\{1 + \beta \cos[(\pi/2)\xi]\} \tag{57}$$

for the beam that is clamped at one end and free at the other, whereas

$$D(\xi) = (4\delta L^3/\pi^2)\{1 + \beta \sin[(\pi/2)\xi]\} \tag{58}$$

with reference to the column that is simply supported at one end and guided at the other. For a column that is clamped at both its ends, the stiffness distribution reads

$$D(\xi) = (\delta L^3/4\pi^2)\{1 + \beta[1 - \cos(2\pi\xi)]\} \tag{59}$$

Each stiffness distribution corresponds to a different distribution of external axial load.

Conclusions

In the paper, closed-form harmonic solutions have been derived for the buckling modes of inhomogeneous columns subjected to harmonic axial load distributions, thus showing that a harmonic function can constitute closed-form buckling modes for inhomogeneous columns. Four cases of harmonically varying buckling modes have been postulated and semi-inverse problems have been solved that result in the distributions of the flexural rigidity compatible to the preselected modes and to specified axial load distributions. In all of the considered cases the closed-form solutions have been obtained for the eigenvalue parameter. The presented methodology of solving semi-inverse problems represents in actuality a design problem against buckling within the class of inhomogeneous columns.

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A. Berman
Associate Editor